BMO – IMO QUALIFICATION TESTS
ROMANIA

Problem 1. Let a_1, a_2, a_3, a_4 be the lengths of the sides of a quadrilateral and s its semi-perimeter. Prove that

$$\sum_{i=1}^{4} \frac{1}{s+a_i} \leq \frac{2}{9} \sum_{1 \leq i < j \leq 4} \frac{1}{\sqrt{(s-a_i)(s-a_j)}}$$

When does equality take place?

Călin Popescu

Problem 2. Let $\mathcal{R}_i$, $i = 1, 2, \ldots, n$, be a finite family of mutually disjoint closed rectangular surfaces whose sides are parallel to the coordinate axes. It is also known that the area of $\mathcal{R} = \bigcup_{i=1}^{n} \mathcal{R}_i$ is at least 4 and the projection onto Ox of their union is an interval.

Prove that $\mathcal{R}$ contains three points which are the vertices of a triangle of area 1.

Dan Ismailescu

Problem 3. Find all injective functions $f : \mathbf{N} \to \mathbf{N}$ such that for each n,

$$f(f(n)) \leq \frac{n + f(n)}{2}.$$

Formulated by Cristinel Mortici

Problem 4. Consider an integer $n \geq 2$ and a disc $\mathcal{D}$ in the complex plane. Prove that for every $z_1, z_2, \ldots, z_n \in \mathcal{D}$, there exists $z \in \mathcal{D}$ such that $z^n = z_1 z_2 \cdots z_n$.

Barbu Berceanu, Dan Schwartz, Dan Marinescu

Problem 5. A disc $\mathcal{D}$ is divided into $2n$ equal sectors, n of them are colored in red and the other n are colored in blue. Starting at an arbitrarily chosen sector we number from 1 to n, in a clockwise order, the red sectors. We proceed in the same way with the blue sectors, but in an anticlockwise order.

Prove that there exists a half-disc of $\mathcal{D}$ which contains all the numbers from 1 to n.

Kvant

Problem 6. Which nonnegative integer values can be reached by the expression

$$\frac{a^2 + ab + b^2}{ab - 1}.$$

Mircea Becheanu

Problem 7. Let a, b, c be integers, b be odd and consider the sequence $x_0 = 4, x_1 = 0, x_2 = 2c, x_3 = 3b$,

$$x_n = a x_{n-4} + b x_{n-3} + c x_{n-2}, \text{ for } n \geq 4.$$

Prove that if p is a prime and m a positive integer, then x_{p^m} is divisible by p.

Călin Popescu

Problem 8. A square $ABCD$ is taken inside a circle γ. Inside the angle opposite to $\angle BAD$ is taken the circle tangent to the productions of the lines AB and AD and internally tangent to γ at A_1. Points B_1, C_1, D_1 are defined in the same way.

Prove that the straight lines AA_1, BB_1, CC_1, DD_1 are concurrent.

Radu Gologan, an ideea from Kvant

Problem 9. Let $n > 1$ be a positive integer and X be a set containing n elements. $A_1, A_2, \ldots, A_{101}$ are subsets of X such that the union of any 50 of them has more than $\frac{50}{51}n$ elements.

Prove that there are of the given subsets such that any two of them have non-void intersection.

Gabriel Dospinescu

Problem 10. Prove that

$$\frac{1}{3^m n}\sum_{k=0}^{m}\binom{3m}{3k}(3n-1)^k$$

is an integer, given n a positive integer and m an odd integer.

Călin Popescu

Problem 11. Let I be the incircle of the triangle ABC and A', B', C' be its tangent points with the sides BC, CA, AB respectively. Lines AA' and BB' intersect at P, lines AC with $A'C'$ at M and $B'C'$ with BC intersect at N. Prove that IP and MN are perpendicular.

Classical result

Problem 12. Let $n \geq 2$ be an integer and $a_1, a_2, \ldots, a_n$ real numbers. Prove that for any non-void subset $S \subset \{1, 2, \ldots, n\}$ the following inequality is true

$$\left(\sum_{i\in S} a_i\right)^2 \leq \sum_{1\leq i\leq j\leq n}(a_i + \cdots + a_j)^2.$$

Gabriel Dospinescu